Invariant Subspaces

and

Lomonosov's Theorem

by

Charles W. Neville, March 1999

1. What Is an Invariant Subspace?

V a vector space

$T : V \longrightarrow V$ a linear operator

W a linear subspace of V is *Invariant* if

$T(W) \subseteq W$

2. What Is a Hyper-Invariant Subspace?

$T : V \longrightarrow V$ a linear operator

W a linear subspace of V is *Hyper-Invariant* if

$$S(W) \subseteq W$$

for every S commuting with T

3. Why Study Invariant Subspaces?

V finite dimensional vector space over $\mathbb{C}$

Recall Jordan Canonical Form for T

$$\begin{pmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 1 & \lambda_1 & 0 & 0 & 0 \\ 0 & 1 & \lambda_1 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & 1 & \lambda_2 \end{pmatrix}$$

(a simple 5×5 case)

Closely related to (hyper) invariant subspaces of T

4. Why Study Invariant Subspaces continued

Hyper Invariant Subspaces for T

$W_1 = \mathrm{Ker}(T - \lambda_1 I)$

$W_2 = \mathrm{Ker}(T - \lambda_1 I)^2 \quad W_1 \subseteq W_2 \subseteq W_3$

$W_3 = \mathrm{Ker}(T - \lambda_1 I)^3$

$W_4 = \mathrm{Ker}(T - \lambda_2 I) \quad W_4 \subseteq W_5$

$W_5 = \mathrm{Ker}(T - \lambda_2 I)^2$

And direct sums of the above

$W_1 \cdots W_5$ basically given by the canonical form basis!

5. Infinite Dimensional Case

V infinite dimensional Banach space

$T : V \longrightarrow V$ a bounded linear operator

W a *closed* subspace of V is *Invariant* if

$T(W) \subseteq W$

6. Famous Open Question

Does Every Bounded Linear Operator on an Infinite Dimensional Banach Space Have a Proper ($\neq \{0\}, \neq V$) (*closed*) Invariant Subspace

Conjecture that this is so is

The Invariant Subspace Conjecture

7. Known Answers

The Known Answers to The Invariant Subspace Question are:

NO for V a complete locally convex space and T continuous – cf. Schaeffer

NO for V a Hilbert space and T a closed linear operator – cf. Halmos

NO (famously) for V a non-reflexive Banach space and T bounded – Read

UNKNOWN for V a reflexive Banach space and T bounded

Ergo UNKNOWN for V a Hilbert space and T bounded

8. More Known Answers

More Known Answers to The Invariant Subspace Question are:

YES if T has a disconnected spectrum

YES (famously) if V is a Hilbert space and T is normal, ie. T commutes with its adjoint. (This is almost the Spectral Theorem)

YES if T commutes with a Compact Operator – Lomonosov's Theorem, the subject of this talk

9. Lomonosov's Theorem

Every bounded operator commuting with a (non-zero) compact operator has a (proper, closed) invariant subspace.

Proof:

T a non-zero compact operator on a Banach space V.

S a bounded operator on V commuting with T.

If S has a proper closed invariant subspace, we are done.

If not, argue by contradiction: Show that S must have a proper closed invariant subspace after all.

10. Lomonosov's Theorem – Proof

T a non-zero compact operator on a Banach space V.

S a bounded operator on V commuting with T, with no proper closed invariant subspace.

F a *non-linear continuous function* constructed from S.

K a *compact convex* subset of V, disjoint from 0, such that

$FT(K) \subseteq K$.

$FT(K) \subseteq K$, and K is compact.

By the Leray-Schauder fixed point theorem, FT has a fixed point e in K,

$FT(e) = e$.

By construction,

$F(T(e)) = \mathcal{P}(S)(T(e))$,

where $\mathcal{P}(S)$ is a polynomial in the operator S.

In summary, we have constructed a polynomial $\mathcal{P}(S)$, such that

$\mathcal{P}(S)T(e) = e$.

12. Lomonosov's Theorem – Proof

$\mathcal{P}(S)T(e) = e$

e is an eigen-vector, with eigen-value 1, for the operator $\mathcal{P}(S)T$.

Let W_1 be the eigen-space corresponding to the eigenvalue 1 for $\mathcal{P}(S)T$,

$W_1 = \{w \in V : \mathcal{P}(S)T(w) = w\}$.

S commutes with $\mathcal{P}(S)$ (because it is a polynomial in S) and with T (by hypothesis), so the eigen-space W_1 is invariant under the action of S.

This contradicts the assumption that S has no proper closed invariant subspace.

13. Lomonosov's Theorem – Proof

In More Detail:

For clarity, let $U = \mathcal{P}(S)T$. Recall that

a. W_1 is the space of eigen-vectors for the operator U corresponding to the eigen-value 1.
b. S commutes with the operator U.

If $w \in W_1$, then $U(w) = w$, so

$S(w) = S(U(w)) = U(S(w))$,

that is w lies in the eigen-space W_1. Thus W_1 is invariant for U.

14. Lomonosov's Theorem – Proof

W_1 is an invariant subspace for the operator S.

Questions:

a. Is W_1 *closed*?

b. Is W_1 different from $\{0\}$?

c. Is W_1 different from V?

Answers:

a. Yes, because the operator U is continuous. ($U(w_n) \to w$ and $U(w_n) = w_n$ implies $U(w) = w$.)

b. Yes, because $e \in W_1$ and $e \neq 0$. (Remember, $e \in K$ and K does not contain 0.)

15. Lomonosov's Theorem – Proof

Questions:

a. Is W_1 *closed*? – Yes.

b. Is W_1 different from $\{0\}$? – Yes.

c. Is W_1 different from V?

Answers:

c. Yes, because $U \neq I$ and I is the only operator for which the eigen-space $W_1 = V$. ($U \neq I$ because $U = \mathcal{P}(S)T$ is *compact*.)

16. Lomonosov's Theorem – Proof

So we started with a non-zero compact operator T and a bounded operator S which commutes with T. We have assumed S has no proper closed invariant subspace, and we have derived a contradiction by producing a proper closed invariant subspace W_1 for S.

All we have to do now is construct the function F, the polynomial $\mathcal{P}(S)$ and the compact set K.

Construction of F:

Let $a \in V$ be such that $T(a) \neq 0$. Let $B = B(a, \rho)$ be a small closed ball centered at a such that $0 \notin T(B)$.

$T(B)$ is *compact* (because T is compact).

17. Construction of F

Suppose the bounded operator S, commmuting with T, has no proper closed invariant subspace. Then for each v in the compact set $T(B)$, the subspace,

$$\{\mathcal{P}(S)(v) : \mathcal{P}(S) \text{ a polynomial in S}\}$$

is dense in V. (In fact, this must be true for every $v \in V$ different from 0, or S would have a proper closed invariant subspace, contrary to assumption.)

For each $v \in T(B)$, let $\mathcal{P}_v(S)$ be a polynomial in S such that $\mathcal{P}_v(S)(v)$ belongs to the interior of B. (We know such a polynomial exists because the above subspace is dense in V and B has non-void interior.)

$\mathcal{P}_v(S)(v)$ belongs to the interior of B, for each $v \in T(B)$.

For each $v \in T(B)$, let B_v be a small open ball centered at v, such that $\mathcal{P}_v(S)(B_v) \subseteq B$. (We know such a ball B_v exists because $\mathcal{P}_v(S)$ is continuous.)

The open balls B_v form an open covering of $T(B)$. Because $T(B)$ is compact, there is a finite open subcovering, which, by abuse of notation, we shall denote by $B_1, \ldots, B_n$.

And for each B_j, there is a polynomial in the operator S, which, by abuse of notation again, we shall denote by $\mathcal{P}_j(S)$, such that $\mathcal{P}_j(B_j) \subseteq B$.

19. Digression – Partitions of Unity

A *finite partition of unity* is a family of continuous, non-negative real valued functions, $f_1, \ldots, f_n$ such that

$\Sigma f_j = 1$

If $U_1, \ldots, U_n$ is a finite open covering of a set X, the partition of unity $f_1, \ldots, f_n$ is *subordinate* to the covering $U_1, \ldots, U_n$ if the support of f_j lies entirely in the set U_j, for each j (if f_j is zero outside of the set U_j, for each j).

Theorem:

If $U_1, \ldots, U_n$ is a finite open covering of a compact set X, then there exists a finite partition of unity subordinate to the covering.

20. Construction of F

The open balls $B_1, \ldots B_n$ form a finite open covering of the compact set $T(B)$, and $\mathcal{P}_j(B_j) \subseteq B$ for each j.

Thus there is a finite partition of unity, $f_1, \ldots, f_n$ subordinate to the open covering $B_1, \ldots, B_n$.

Let $F(x) = \Sigma f_j(x)\mathcal{P}_j(S)(x)$, for each $x \in T(B)$.

The function F is not linear (because of the f_j multipliers), but it is continuous, and $F(T(B) \subseteq B$.

All that is left to do now is to construct the polynomial $\mathcal{P}(S)$ and the compact set K.

21. Construction of $\mathcal{P}(S)$

The construction of the polynomial $\mathcal{P}(S)$ is easy. Recall

$F(x) = \Sigma f_j(x)\mathcal{P}_j(S)(x)$.
e is a fixed point for FT, $FT(e) = e$
We want $FT(e) = \mathcal{P}(S)(e)$

Simply set

$\mathcal{P}(S)(x) = \Sigma f_j(e)\mathcal{P}_j(S)(x)$.

22. Construction of K

First Try: Let K be the closed convex hull of $F(T(B))$.

Problem: $F(T(B))$ is compact alright, being the continuous image of a compact set, but, unfortunately, the closed convex hull of a compact set is not necessarily compact. So we have to try something else.

Second Try: Recall the operator polynomials $\mathcal{P}_j(S)$. Recall that $\mathcal{P}_j(S)(B_j) \subseteq B$. This relation still holds true if the open balls B_j are replaced with their closures. By abuse of notation, denote the closures by B_j.

23. Construction of K

The balls B_j are now closed.

Let $K_j = \mathcal{P}_j(S)(B_j \cap T(B))$.

Each K_j is convex and compact. Let K be the closed convex hull of the union of the K_j's. Then

a. K is compact because the closed convex hull of a finite union of compact convex subsets is compact.
b. $K \subseteq B$ because each $K_j \subseteq B$, and B is closed convex.
c. $FT(K) \subseteq K$ because $F(T(B)) \subseteq K$ by the construction of F.

QED

24. Elements of the Proof

a. *Non-linearization* of the problem,

$$F(x) = \Sigma f_j(x)\mathcal{P}_j(S)(x).$$

b. Use of partitions of unity.

c. Use of the Leray – Schauder fixed point theorem for compact convex sets.

25. Generalizations of Lomonosov's Theorem

Theorem: Every operator which commutes with a (nonzero) compact operator has a (proper closed) hyperinvariant subspace. – Pearcy and Rovnyak, cf. Radjavi and Rosenthal.

The Leray – Schauder fixed point theorem applies to compact convex subsets in a *locally convex space*, not just to such sets in a Banach space. In a reflexive Banach space, the closed ball B is weakly compact. Many people have tried to apply Lomonosov's argument in this case to prove the Invariant Subspace Conjecture for reflexive Banach spaces, always without success.

26. References

P. Halmos, *A Hilbert Space Problem Book*, Van Nostrand, 1967.

H. Radjavi and P. Rosenthal, *Invariant Subspaces*, Springer, 1973.

C. J. Read, "A short proof converning the invariant subspace problem," *J. London Math. Soc.* (2) 34 (1986), 335–348. (for ℓ^1.)

C. J. Read, "The invariant subspace problem on some Banach spaces with separable dual," *Proc. London Math. Soc.* (3) 58 (1989), 583 - 607.

H. Schaeffer, *Locally Convex Spaces*