New Results on Primes from an Old Proof of Euler's

by

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1. Introduction.

S an infinite set of positive integers with $0 \notin S$.

S has polynomial density α if

$$\operatorname{card}\{s \in S : s \leq n\} \geq K n^{1/\alpha},$$

where K is a constant > 0.

Equivalently, has polynomial density α if

> S is the range of a strictly monotone increasing positive integer valued sequence $(s_n)_{n=1}^{\infty}$, such that $s_n = O(n^{\alpha})$.

Note: α might not be in integer, and $\alpha \geq 1$.

$P(S)$ = the set of prime factors of elements of S,

$$P(S) = \{p : p \text{ is prime and } p \,|\, s \text{ for at least one } s \in S\}.$$

$\pi_S(n)$ is the number of prime factors $\leq n$ of the elements of S.

Equivalently,

$$\pi_S(n) = \text{card}\{p \in P(S) : p \leq n\}.$$

Clearly $\pi_S(n)$ is the familiar "prime number function" $\pi(n)$ relativized to S.

QUESTION.

What is the distribution of the set $P(S)$ of prime factors of the elements of S?

BACKGROUND.

CASE $S = \mathbb{N}$, the positive integers. In this case, $P(S)$ simplifies to the set of primes.

Euclid (circa 300 BC). $P(S)$ is infinite.

Euler (1737). $P(S)$ is infinite – proof uses infinite products and infinite series, first appearance of the Riemann ζ function.

Historical note: Euler didn't actually present his work as a proof of Euclid's theorem. Rather, he derived such striking new results (for the time) as $\sum 1/p$ diverges, where p ranges over the prime numbers.

CASE $S = \mathbb{N}$, the positive integers.

Tschebyshev (1850).

$$mn/\log n \leq \pi_S(n) \leq Mn/\log n,$$

for all $n \geq 2$ and some suitably chosen constants $0 < m \leq M < \infty$.

Hadamard and de la Vallée Poussin (independently, 1896).

$$\lim_{n \to \infty} \pi_S(n)/(n/\log n) = 1.$$

CASE $S =$ an arithmetic progression,

$$S = \{an + b : n = 1, 2, 3, \ldots\}$$

Now, $P(S)$ is strictly greater than the number of primes in S.

Dirichlet (1837). If a and b are relatively prime, the number of primes in S is infinite – proof introduces Dirichlet L functions.

We are still looking for a *general*, *simple* elementary proof.

Furstenberg's Proof.

Furstenberg (1955). A *topological* proof of Euclid's theorem that P(S) is infinite if S is the set of positive integers!

Furstenberg's proof is an important beginning example in the theory of profinite groups, and Dirichlet's theorem, which asserts the existence of infinitely many primes rather than prime factors, has a simple interpretation in terms of profinite groups. (A. Lubotzky, *Bull. Amer. Math. Soc.* **38** (4), 2001, pp. 475–479.)

Could this lead to a new simple *topological* proof of Dirichlet's theorem? (A. Lubotzky, *ibid.*)

CASE $S =$ the integer range of a non-constant polynomial,

$$S = \{|F(n)| : n = 1, 2, 3, \ldots, F(n) \neq 0\},$$

where $F(n)$ is a non-constant polynomial with integer coefficients.

Again, $P(S)$ is strictly greater than the number of primes in S.

Buniakowski (1857). *Conjectured* if there is no common factor of all the elements of S, then S contains infinitely many primes.

Status of the Buniakowski Conjecture

One of the great unsolved questions of number theory – even unknown if there are infinitely many primes of the form $n^2 + 1$! (M. Ram Murty, *Amer. Math. Monthly* **109**(5) (2002), 452-458, H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, 5th ed., 1993.)

Folk Theorem. $P(S)$ is infinite.

Gotchev (2001). $P(S)$ is infinite – a *topological* proof!

CASE $S =$ an arbitrary set of polynomial density.

Recall S is a set of polynomial density if it is the range of a strictly monotone increasing positive integer valued sequence of polynomial growth.

Neville (2002). Quantitative estimates on $\pi_S(n)$. Recall $\pi_S(n)$ is the number of prime factors $\leq n$ of the elements of S.

Neville (2002). Quantitative estimates for arbitrary sets of polynomial density, not just integer ranges of polynomials.

Theorem 1. Let S be a set of polynomial density α.

(a) $\sum_{p\in P(S)} p^{-1/\alpha} = \infty$.

(b) $\sum_{n=1}^{\infty} \pi_S(n)/n^{1+1/\alpha} = \infty$.

(c) If $\sum_{n=1}^{\infty} a_n < \infty$, where $a_n > 0$, then $\pi_S(n)/n^{1+1/\alpha} \geq a_n$ infinitely often.

Corollary 1.

(a) The set $P(S)$ is infinite.

(b) Let $r > 1$. Then

$\pi_S(n) \geq n^{1/\alpha}/(\log n)^r$ infinitely often.

$\pi_S(n) \geq n^{1/\alpha}/\log n(\log\log n)^r$ infinitely often.

etc.

Proof of Corollary 1.

(a) – use theorem 1a.

(b) – this is equivalent to

$\pi_S(n)/n^{1+1/\alpha} \geq 1/n\,(\log n)^r$ infinitely often,

$\pi_S(n)/n^{1+1/\alpha} \geq 1/n\ \log n\,(\log\log n)^r$ infinitely often,

etc.,

and the series

$\sum_{n=1}^{\infty} 1/n\,(\log n)^r$,

$\sum_{n=1}^{\infty} 1/n\ \log n\,(\log\log n)^r$,

etc.,

are all convergent. Use theorem 1c.

Proof of Theorem 1a.

Proof – a simple modification of Euler's famous 1737 proof by infinite series of Euclid's theorem on the infinitude of primes.

Consider the (possibly) infinite product

$$\prod_{p\in P(S)} 1/(1-p^{-1/\alpha}).$$

Show the product diverges. So let N be a large positive integer. Consider the partial product,

$$\prod_{p\in P(S),p\leq N} 1/(1-p^{-1/\alpha}) = \prod_{p\in P(S),p\leq N} (1+p^{-1/\alpha}+p^{-2/\alpha}+\cdots)$$
$$= \sum_{s\in S,s\leq N} s^{-1/\alpha} + \text{ other positive terms.}$$

Recall: Another way of writing S is $S = \{s_1, s_2, s_3, \ldots\}$, where $(s_j)_{j=1}^{\infty}$ is a strictly monotone increasing sequence of positive integers such that $s_j \leq Kj^{\alpha}$, where $K > 0$ is a constant.

Thus,

$$\sum_{s \in S, s \leq N} s^{-1/\alpha} = \sum_{j=1}^{n} s_j^{-1/\alpha} \geq \sum_{j=1}^{n} K^{-1/\alpha}(j^{\alpha})^{-1/\alpha} = K^{-1/\alpha} \sum_{j=1}^{n} j^{-1}.$$

(Here, n is the largest integer such that $s_n \leq N$.)

$\sum_{j=1}^{\infty} j^{-1}$ diverges. Thus $\prod_{p \in P(S)} 1/(1 - p^{-1/\alpha})$ diverges.

Recall: An infinite product $\prod_{n=1}^{\infty}(1+a_n)$, with $a_n \geq 0$, diverges $\iff$ the infinite series $\sum_{n=1}^{\infty} a_n$ diverges.

Thus $\prod_{p\in P(S)} 1/(1-p^{-1/\alpha})$ diverges $\implies \sum_{p\in P(S)} \big(1/(1-p^{-1/\alpha})-1\big)$ diverges.

The series $\sum_{p\in P(S)} \big(1/(1-p^{-1/\alpha})-1\big)$ is, as we shall see, term-by-term comparable with the series $\sum_{p\in P(S)} p^{-1/\alpha}$.

Thus $\sum_{p\in P(S)} p^{-1/\alpha}$ diverges. This proves theorem 1a, provided we prove the comparability assertion.

The Two Series Are Comparable.

Note $p^{-1/\alpha} \leq 2^{-1/\alpha}$. Apply the familiar estimates,

$$\begin{aligned} 1 + p^{-1/\alpha} &< 1 + p^{-1/\alpha} + p^{-2/\alpha} + \cdots \\ &= 1/(1 - p^{-1/\alpha}) = 1 + p^{-1/\alpha}/(1 - p^{-1/\alpha}) \\ &\leq 1 + p^{-1/\alpha}/(1 - 2^{-1/\alpha}). \end{aligned}$$

Conclude that

$$p^{-1/\alpha} < 1/(1 - p^{-1/\alpha}) - 1 \leq p^{-1/\alpha}/(1 - 2^{-1/\alpha}).$$

This proves proves the two series are comparable, and so completes the proof of theorem 1a.

Proof of Theorem 1b.

$$\sum_{p \in P(S)} p^{-1/\alpha} = \lim_{N \to \infty} \int_1^N t^{-1/\alpha}\, d\pi_S(t),$$

where the right hand side is interpreted as a Stieltjes integral.

Integrate by parts to obtain

$$\sum_{p \in P(S)} p^{-1/\alpha} = \lim_{N \to \infty} \left[N^{-1/\alpha} \pi_S(N) + (1 + 1/\alpha) \int_1^N \pi_S(t) t^{-(1+1/\alpha)}\, dt \right].$$

(The integration by parts is justified because the integrand $t^{-1/\alpha}$ is continuous.)

By theorem 1a, the left hand series diverges to infinity. We could immediately conclude that

$$\lim_{N\to\infty} \int_1^N \pi_S(t) t^{-(1+1/\alpha)}\, dt = \infty,$$

if it were not for the troublesome first term on the right, $N^{-1/\alpha}\pi_S(N)$. To handle this term, divide the argument into two cases.

CASE 1 of troublesome first term on right.

Suppose ∃ a constant $K > 0$ such that $N^{-1/\alpha}\pi_S(N) < K$, for infinitely many positive integers N, say $N_1, N_2, N_3, \ldots$. Then

$$\lim_{j\to\infty} \int_1^{N_j} \pi_S(t)t^{-(1+1/\alpha)}\, dt = \infty.$$

Because $\int_1^N \pi_S(t)t^{-(1+1/\alpha)}\, dt$ is an increasing function of N,

$$\lim_{N\to\infty} \int_1^{N} \pi_S(t)t^{-(1+1/\alpha)}\, dt = \infty.$$

CASE 2 of troublesome first term on right.

Suppose there is no such constant K. Then for every $K > 0$, $N^{-1/\alpha}\pi_S(N)$ is eventually $> K$.

Choose any such $K > 0$, choose N_K so that $N^{-1/\alpha}\pi_S(N) > K$ for all (real) $N > N_K$. Use the simple estimate,

$$\int_{N_K}^{N} \pi_S(t)t^{-(1+1/\alpha)}\,dt > K\int_{N_K}^{N} t^{-1}\,dt.$$

Conclude

$$\lim_{N\to\infty} \int_1^N \pi_S(t) t^{-(1+1/\alpha)}\, dt = \infty.$$

By the integral test (or at least by its proof),

$$\sum_{n=1}^{\infty} \pi_S(n) n^{-(1+1/\alpha)} = \infty.$$

This completes the proof of theorem 1b.

Proof of Theorem 1c.

Use theorem 1b.

Example 1. *Polynomial density is needed.* Consider the non-polynomially dense set $S = \{2^n : n = 1, 2, 3, \ldots\}$. Then $P(S) = \{2\}$.

Example 2. *The exponent $1/\alpha$ is the best possible.* Let α be an integer ≥ 3. Use a theorem of Iwaniec and Piutz (*Monatshefte f. Math.* **98** (1984)) to show there is a prime p_n with $n^\alpha < p_n \leq (n+1)^\alpha$ for all $n \geq$ some integer N_α.

Let $S = \{p_n : n = N_\alpha, N_\alpha + 1, N_\alpha + 2, \ldots\}$. S has polynomial density α, p_n is asymptotically equal to n^α, and $\pi_S(n)$ is asymptotically equal to $n^{1/\alpha}$.

Example 3. *The* $\log n$ *denominator is needed.* Let $S = \mathbb{N}$, the set of positive integers. Use Tschebyshev's inequalities.

Open Questions.

Question 1. Is there an example where both the exponent $1/\alpha$ and the denominator $\log n$ appear at the same time?

Question 2. There cannot be a prime number theorem for $P(S)$, or even a pair of Tschebyshev inequalities (see example 2). But can corollary 1 be improved to a *one-sided* Tschebyshev inequality? In other words, for every set S of polynomial density α, is it true that

$$m_S\, n^{1/\alpha}/\log n \le \pi_S(n),$$

for all $n \ge 2$ and some suitably chosen constant $m_S > 0$ (depending on S)?

Web References

This handout – http://www.cwnresearch.com/research/

The complete paper –

http://www.cwnresearch.com/research/

http://arXiv.org/abs/math.NT/0210282

Credits.

No MICROSOFT software was used in the making of this talk.

No MONSTERS were hurt in the making of this talk.